SS 2016

Dr. Milena Pabiniak
Marc Kegel

Differential Geometry

Homework 8

Mandatory Exercise 1. (10 points)
Consider the two embeddins of $\phi_1 \colon T^2 \to \mathbb{R}^4$ and $\phi_2 \colon T^2 \to \mathbb{R}^3$, locally given by

$$\phi_1(\alpha, \beta) = (\cos\alpha, \sin\alpha, \cos\beta, \sin\beta),$$
$$\phi_2(\alpha, \beta) = ((2+\cos\alpha)\cos\beta, (2+\cos\alpha)\sin\beta, \sin\alpha).$$

Using these two embeddings one obtains two metrics on T^2, induced from the metrics from $\mathbb{R}^4$ and $\mathbb{R}^3$ respectively. Compare $[\frac{\partial}{\partial\alpha}, \frac{\partial}{\partial\beta}]$ and $\nabla_{\frac{\partial}{\partial\alpha}} \frac{\partial}{\partial\beta}$ in these two metrics.

Mandatory Exercise 2. (10 points)
Let G be a Lie group. Recall that a **bi-invariant** metric is a Riemannian metric for which left and right translations are isometries.

(a) Show that the existence of a bi-invariant metric on G is equivalent to the existence of an $Ad(G)$-invariant scalar product on T_eG. (Recall that any compact Lie group has a bi-invariant metric. This does not need to hold for non-compact groups, for example $SL(n, \mathbb{R})$.)

(b) From now on let G be endowed with a bi-inavriant Riemannian metric and denote by $i \colon G \to G$ the smooth map defined by $i(g) = g^{-1}$. Compute $(di)_e$ and conclude that i is an isometry.

(c) Let $v \in \mathfrak{g} = T_eG$ and c_v be the geodesic satisfying $c_v(0) = e$ and $\dot{c}_v(0) = v$. Show that if t is sufficently small then $c_v(-t) = (c_v(t))^{-1}$. Conclude that c_v is defined on $\mathbb{R}$ and satisfies $c_v(t+s) = c_v(t) \cdot c_v(s)$ for all $t, s \in \mathbb{R}$.

(d) Use (c) to prove that the geodesics on G are the integral curves of all left invariant vector fields.

(e) Let ∇ be the Levi-Civita connection on G associated with a biinvariant metric. Show that

$$\nabla_X Y = \frac{1}{2}[X, Y],$$

for any two left invariant vector fields X and Y.

Suggested Exercise 1. (0 points)
Let ∇ be the Levi-Civita connection on $\mathbb{R}^2 \setminus \{0\}$ (induced from $\mathbb{R}^2$). Compute $\nabla_X Y$ and $\nabla_Y X$ for $X_{(x,y)} = (-y, x)$ and $Y_{(x,y)} = \frac{1}{r}(x, y)$.

Suggested Exercise 2. (0 points)
Let g be a Riemannian metric on M and $\tilde{g} = f^2 g$ for some nowhere vanishing smooth function f on M. Give the relation between the Levi-Civita connection ∇ associated to g and the Levi-Civita connection $\tilde{\nabla}$ associated to $\tilde{g}$.

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Suggested Exercise 3. (0 points)
Show that on $S^n \subset \mathbb{R}^{n+1}$, with the induced metric, the vector fields

$$x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i}$$

$(0 \leq i, j \leq n)$, are Killing vector fields.

Suggested Exercise 4. (0 points)
A smooth function $f \in C^\infty(M)$ defines a $(0,2)$ tensor field $Hess(f)$ on M, called the **Hessian** of f, by

$$(X, Y) \to (\nabla_X df)(Y),$$

for X, Y - vector fields on M.

(a) Show that it is indeed a tensor and that it is symmetric.

(b) Give the expression of it in a local chart.

(c) Show that if $S^n \subset \mathbb{R}^{n+1}$ is given the induced metric, and if f is the restriction to S^n of a linear form, then the Hessian is a multiple of the metric g: $Hess(f) = -fg$.

Suggested Exercise 5. (0 points)
Let (M, g) and $(N, \tilde{g})$ be isometric Riemannian manifolds with Levi-Civita connections ∇ and $\tilde{\nabla}$, and let $f\colon M \to N$ be an isometry. Show that:

(a) $f_* \nabla_X Y = \tilde{\nabla}_{f_* X} f_* Y$ for all vector fields X, Y on M.

(b) If $c\colon I \to M$ is a geodesic then $f \circ c\colon I \to N$ is also a geodesic.

Suggested Exercise 6. (0 points)
Let $f\colon M \to \mathbb{R}$ be a smooth function on a Riemannian manifold with $||\operatorname{grad}(f)|| = 1$. Show that the integral curves of $\operatorname{grad}(f)$ are geodesics.

Suggested Exercise 7. (0 points)
Let $f\colon M \to N$ be an isometry whose set of fixed points is a connected 1-dimensional submanifold $N \subset M$. Show that N is the image of a geodesic.

Suggested Exercise 8. (0 points)
Show that X is a Killing vector field if and only if the local flow of X consists of local isometries of (M, g).

Hand in: Monday 13th June
in the exercise session
in Seminar room 2, MI